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THE STABILITY OF SATURN'S RINGS

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Maxwell determined the conditions of stability of a single ring of small particles moving round a large primary. He also made some incomplete remarks on the effects of introducing a second ring. The present investigation considers in greater detail the stability of two rings of particles moving about a primary and subject to the gravitational attractions of the primary and of each other. It is shown that such a system, under conditions satisfied by the Saturnian system, is stable, the particles oscillating finitely about their mean positions. It is inferred that the Saturnian system, considered as a number of such rings, is therefore also stable.

Introduction

In his paper on the effects of collisions among the particles of Saturn's ring, Jeffreys (1947) has deduced that the ultimate state would be a ring nowhere more than one particle thick and with spacing just sufficient to avoid collisions. The question then arises whether, in this state, the system would be stable or unstable under small disturbances.

Maxwell, in his memoir on Saturn's Rings (1857), showed that a single ring of equal particles, moving about the planet with a velocity equal to that of a particle in a circular orbit at the same distance, would be stable under small disturbances if the particles forming the ring were sufficiently small in mass compared with that of Saturn. In paragraph 25 of his paper he further examined the motion of two such rings reacting upon each other. This part of his work is, however, incomplete and lacking in generality. The present paper examines Maxwell's problem of two rings afresh.

Two co-planar circles, concentric with Saturn, have equal particles approximately equally spaced upon them; and the particles are subject to the gravitational attraction of Saturn and of each other. The number of particles in each ring is assumed to be the same, although an extension of the analysis could be made so as to allow for the numbers being different. It is found that when the particles are sufficiently small and sufficiently numerous, there exists a system of steady motions in which the particles are exactly equally spaced on their respective rings and the rings rotate about the primary with appropriate velocities.

Small displacements are then given to the particles in the plane of motion, and it is deduced that the motion can be expressed in terms of series of periodic functions. The motion, in the sense of small displacements, is therefore stable. The general conclusion is that, under conditions satisfied by the Saturnian rings, a pair of rings forms a stable system.

The complete problem of a number of rings could be attacked by an extension of the methods used for a pair of rings, though the work would be lengthy. From the indications of the analysis here given however, it would be safe to conclude that, just as for two rings, stability would ensue for any finite number of rings, under the stated conditions.

When the number of particles in a ring is taken as p, the equations of stability appear as $4p$ linear differential equations of the second order. By means of a Poincaré transformation, these are reduced to four. The relative simplicity of the coefficients appearing in these equations follows from the large value of p, which enables asymptotic approximations to be used.

The equations themselves have periodic coefficients and involve a small parameter. By the methods of Poincaré (1892) and Moulton (1920) solutions in convergent series of powers of this parameter (or of its square root) are readily obtained.

Formulation of the problem

Consider the primary, Saturn, as a fixed Newtonian centre of mass M. With centres at Saturn let there be two approximately circular co-planar systems of p particles, each of mass m, one circle being of radius a and the other of radius a', with $a' > a$. The particles are subject to the gravitational attraction of M and of each other. If the mutual attraction of the particles is neglected, a possible system would be for each set to be on the circumference of its own circle at the vertices of an inscribed regular polygon. The actual motion of the particles involves a departure from those mean positions.

The distances apart of pairs of consecutive mean positions on the first circle will be $2\pi a/p$, and on the second circle $2\pi a'/p$. Now choose the radii so that $a'-a = \nu\pi(a+a')/p$, where ν is some positive number. We have no knowledge as to the relation in the actual Saturnian system between the distances apart of the consecutive particles in one ring and the differences of the radii of that ring and the next. But by use of the factor ν, adjustments could be made to meet any value of this relationship.

Putting $a/a' = \alpha$, which is less than unity, we have

$$\alpha = \frac{1-\nu\pi/p}{1+\nu\pi/p}.$$

We shall assume, consistently with observation, that the number of particles in any one ring is very large.

Then
$$\alpha = 1-2\nu\pi/p$$
and
$$\lim_{p\to\infty} \alpha^p = e^{-2\nu\pi}.$$

The numerical values of this limit are important in what follows:

$$\begin{aligned} \nu = 1, &\quad \alpha^p \to 1{\cdot}867 \times 10^{-3}, \\ = 2, &\quad\quad 3{\cdot}49 \times 10^{-6}, \\ = 3, &\quad\quad 6{\cdot}54 \times 10^{-9}. \end{aligned}$$

We also require the Laplace expansions

$$(1-2\alpha\cos\phi+\alpha^2)^{-\frac{1}{2}} = \tfrac{1}{2}b_0 + \sum_1^\infty b_n \cos n\phi,$$

$$(1-2\alpha\cos\phi+\alpha^2)^{-\frac{3}{2}} = \tfrac{1}{2}c_0 + \sum_1^\infty c_n \cos n\phi,$$

where
$$\tfrac{1}{2}b_n = \frac{\Gamma(n+\frac{1}{2})\,\alpha^n}{\Gamma(\frac{1}{2})\,\Gamma(n+1)} F(\tfrac{1}{2}, n+\tfrac{1}{2}, n+1, \alpha^2),$$

$$\tfrac{1}{2}c_n = \frac{\Gamma(n+\frac{3}{2})\,\alpha^n}{\Gamma(\frac{3}{2})\,\Gamma(n+1)} F(\tfrac{3}{2}, n+\tfrac{3}{2}, n+1, \alpha^2). \tag{1}$$

Using the well-known result, that if $\gamma-\alpha-\beta<0$, then, as $x\to 1-0$,

$$F(\alpha,\beta,\gamma,x)\div\left\{\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}(1-x)^{\gamma-\alpha-\beta}\right\}\to 1,$$

and applying it to (1), we have

$$\tfrac{1}{2}c_n\to\frac{4}{\pi}\frac{\alpha^n}{(1-\alpha^2)^2},\quad \text{as}\quad \alpha\to 1.$$

Hence
$$\tfrac{1}{2}c_p=\frac{p^2\,\mathrm{e}^{-2\nu\pi}}{4\nu^2\pi^3},$$

and
$$\tfrac{1}{2}c_{2p}=\tfrac{1}{2}c_p\times 4\,\mathrm{e}^{-2\nu\pi}.$$

We shall therefore neglect c_{2p}, c_{3p}, ... compared with c_p.

Maxwell's criterion for the stability of a single ring was that the mass of the ring should be sufficiently small compared with that of Saturn. In our notation the numerical condition (Tisserand 1891) may be expressed $m/M<2/p^3$. We shall assume in what follows that this condition is fulfilled.

The possibility of steady motion

Taking an origin at the centre of Saturn (assumed fixed), let the plane polar co-ordinates of a particle of the inner ring at time t be $(r_\lambda, \theta_\lambda)$, and those of a particle of the outer ring be $(r'_\lambda, \theta'_\lambda)$, where $\lambda = 1, 2, 3, \ldots, p$.

The equations of motion for a particle of the inner ring are

$$\ddot{r}_\lambda-r_\lambda\dot{\theta}_\lambda^2=-\frac{M}{r_\lambda^2}+\frac{\partial F_\lambda}{\partial r_\lambda}+\frac{\partial G_{\lambda\mu}}{\partial r_\lambda},\tag{2}$$

$$\frac{d}{dt}(r_\lambda^2\dot{\theta}_\lambda)=\frac{\partial F_\lambda}{\partial \theta_\lambda}+\frac{\partial G_{\lambda\mu}}{\partial \theta_\lambda},\tag{3}$$

$$F_\lambda=\sum_\mu{}' m/\Delta_{\lambda\mu},$$

$$\Delta_{\lambda\mu}^2=r_\lambda^2+r_\mu^2-2r_\lambda r_\mu\cos(\theta_\lambda-\theta_\mu),$$

$$G_{\lambda\mu}=\sum_\mu m/D_{\lambda\mu},$$

$$D_{\lambda\mu}^2=r_\lambda^2+r'^2_\mu-2r_\lambda r'_\mu\cos(\theta_\lambda-\theta'_\mu).$$

The summation sign Σ' omits the term where $\lambda=\mu$.

The problem throughout is a plane problem, and no displacements perpendicular to the common plane of the two rings are considered.

Consider the possibility of a solution in which

$$\left.\begin{aligned} r_\lambda&=a, & \theta_\lambda&=\omega t+2\pi\lambda/p,\\ r'_\lambda&=a', & \theta'_\lambda&=\omega' t+2\pi\lambda/p+\epsilon,\end{aligned}\right\}\quad\text{for all }\lambda,$$

and for which ω, ω' are as yet undetermined. Substitution in equation (2) of these values gives

$$-\omega^2=-\frac{M}{a^3}-\frac{m}{a^3}\sum_\mu{}'\tfrac{1}{4}\operatorname{cosec}\pi(\lambda-\mu)/p-\frac{m}{a}\sum_{\mu=1}^{p}\frac{a-a'\cos\{(\omega-\omega')\,t+2\pi(\lambda-\mu)/p-\epsilon\}}{(D_{\lambda\mu}^3)_0}.\tag{4}$$

Writing $\phi = (\omega - \omega')\,t + 2\pi(\lambda-\mu)/p - \epsilon$, the last term gives

$$-\frac{m}{aa'^2}\sum_{\mu}(\alpha-\cos\phi)\,(\tfrac{1}{2}c_0+\sum c_n\cos n\phi) = -\frac{m}{2aa'^2}\sum_{\mu}\left[\alpha c_0 - c_1 + \sum_{n=1}^{\infty}(2\alpha c_n - c_{n-1} - c_{n+1})\cos n\phi\right].$$

Now
$$\sum_{\mu}\cos n\phi = \sum_{\mu}\cos n\{(\omega-\omega')\,t + 2\pi(\lambda-\mu)/p - \epsilon\}$$
$$= 0, \quad \text{unless} \quad n = p,\ 2p,\ 3p,\ \ldots$$
$$= p\cos p\{(\omega-\omega')\,t - \epsilon\},$$
taking only the value $n=p$.

The final value of the summation is then

$$-\frac{mp}{2aa'^2}[\alpha c_0 - c_1 + (2\alpha c_p - c_{p-1} - c_{p+1})\cos p\{(\omega-\omega')\,t-\epsilon\}]. \tag{5}$$

The Laplace coefficients c_n are monotonic functions of α increasing with α in the range $0<\alpha<1$, Further, as $n\to\infty$, $c_n/c_{n+1}\to\alpha$.

Hence for p large, we have approximately

$$2\alpha c_p - c_{p-1} - c_{p+1} = -\frac{4\alpha^{p-1}}{\pi(1-\alpha^2)} = \frac{p\alpha^{p-1}}{\nu\pi^2}$$

and
$$\alpha c_0 - c_1 = -\frac{p\alpha}{\nu\pi^2}.$$

The value of the summation (5) then becomes

$$\frac{mp^2}{2aa'^2}\left[\frac{\alpha}{\nu\pi^2}+\frac{\alpha^{p-1}}{\nu\pi^2}\cos p\{(\omega-\omega')\,t-\epsilon\}\right]. \tag{6}$$

The coefficient of the trigonometric term

$$= \frac{mp^2\alpha^{p-1}}{2aa'^2\nu\pi^2} = \frac{mp^2\alpha^{p+1}}{2a^3\nu\pi^2} < \frac{M\alpha^{p+1}}{a^3p\nu\pi^2} = \frac{M}{a^3}1{\cdot}86\times10^{-10},$$

if p be taken as 10^6 and ν as 1.

The coefficient will diminish still more rapidly if the separation between the rings be made greater, i.e. if ν be increased. The variable term may therefore certainly be neglected compared with M/a^3.

Also, taking now the first term of (6),

$$\frac{mp^2\alpha}{2aa'^2\nu\pi} < \frac{M\alpha^3}{a^3p\nu\pi^2} = \frac{M}{a^3}10^{-7}, \quad \text{for} \quad p = 10^6,\ \nu = 1,\ \alpha = 1.$$

Hence this term adds a very small constant correction to the first term of the right-hand member of (4).

The term
$$\frac{m}{a^3}\sum_{\mu}{}'\,\tfrac{1}{4}\operatorname{cosec}\pi(\lambda-\mu)/p = \frac{m}{4a^3}\sum_{n=1}^{p-1}\operatorname{cosec} n\pi/p$$
$$= \frac{mp}{8a^3\pi}\log(2p^2/\pi^2), \quad \text{approximately},$$
$$< \frac{M}{a^3}\frac{1}{4\pi p^2}\log(2p^2/\pi^2).$$

This term is again quite negligible compared with M/a^3.

The equation (4) then gives with a high degree of approximation $\omega^2 = M/a^3$.

In the same way the equation (3) gives approximately

$$\frac{d}{dt}(r_\lambda^2\dot{\theta}_\lambda) = -\sum_\mu{}' \frac{m\cos(\lambda-\mu)\pi/p}{4a\sin^2(\lambda-\mu)\,\pi/p} - \sum_\mu \frac{maa'\sin\phi}{(D^3_{\lambda\mu})_0}. \tag{7}$$

The first term on the right vanishes in summation. Also

$$\sum_\mu \frac{\sin\phi}{(D^3_{\lambda\mu})_0} = \sum_\mu \frac{\sin\phi}{a'^3}\{\tfrac{1}{2}c_0 + \sum c_n\cos n\phi\}$$

$$= \sum_\mu \frac{1}{2a'^3}[c_0\sin\phi + \sum c_n\{\sin(n+1)\,\phi - \sin(n-1)\,\phi\}]$$

$$= \frac{p}{2a'^3}(c_{p-1}-c_{p+1})\sin p\{(\omega-\omega')\,t-\epsilon\}.$$

As before

$$\tfrac{1}{2}(c_{p-1}-c_{p+1}) \to \frac{4\alpha^{p-1}}{\pi(1-\alpha^2)}$$

and the equation (7) becomes

$$\frac{d}{dt}(r_\lambda^2\dot{\theta}_\lambda) = -\frac{4map\alpha^{p-1}}{\pi a'^2(1-\alpha^2)}\sin p\{(\omega-\omega')\,t-\epsilon\}.$$

Therefore

$$r^2\dot{\theta}_\lambda = a^2\omega + \frac{4ma\alpha^{p-1}}{\pi a'^2(1-\alpha^2)\,(\omega-\omega')}\cos p\{(\omega-\omega')\,t-\epsilon\}.$$

But $\omega-\omega' = 3\nu\pi\omega/p$, approximately, hence

$$\frac{4ma\alpha^{p-1}}{\pi a'^2(1-\alpha^2)\,(\omega-\omega')} = \frac{p^2ma\alpha^{p-1}}{3\nu^2\pi^3\omega a'^2} < \frac{2\omega a^2\alpha^{p+1}}{3p\nu^2\pi^3} = 4\times10^{-11}a^2\omega, \quad \text{for} \quad \nu = 1,\, p = 10^6.$$

The integral of this equation may therefore be taken as

$$r_\lambda^2\dot{\theta}_\lambda = a^2\omega.$$

It has therefore been shown that the form of solution assumed satisfies the equations (2) and (3) with a high degree of accuracy when p is large.

Similarly for the second ring we have

$$a'^3\omega'^2 = M.$$

These two results show that when p is large the particles in each ring remain at the corners of regular inscribed polygons, even when the rings are nearly equal in radius, and that the rings rotate as single particles would at the same distance.

Equations for the disturbed motion

We assume that the steady motion of the last section is now disturbed slightly. We take

$$r_\lambda = a(1+\rho_\lambda), \quad \theta_\lambda = \omega t + 2\pi\lambda/p + \eta_\lambda,$$
$$r'_\lambda = a'(1+\rho'_\lambda), \quad \theta'_\lambda = \omega' t + 2\pi\lambda/p + \epsilon + \eta'_\lambda.$$

These are substituted in equations (2) and (3) and only the linear terms in ρ, η retained. The terms corresponding to (4) and (7) cancel out identically. The result is a set of two homogeneous linear equations. Two more parallel equations can be written down for the particles in the outer ring. The four equations are to be solved, giving the values of ρ_λ, ρ'_λ, η_λ, η'_λ in terms of the time and of arbitrary constants. In giving λ values 1 to p we arrive at $4p$ equations defining the motion of the $2p$ particles forming the two rings.

Following this out we have the four equations

$$\begin{aligned}
\ddot\rho_\lambda-2\omega\dot\eta_\lambda-3\omega^2\rho_\lambda &= \sum_\mu{}'\left[\left\{\frac{\partial^2}{\partial r_\lambda^2}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0\rho_\lambda+\left\{\frac{\partial^2}{\partial r_\lambda\,\partial r_\mu}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0\rho_\mu+\frac{1}{a}\left\{\frac{\partial^2}{\partial r_\lambda\,\partial\theta_\lambda}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta_\mu)\right]\\
&\quad+\sum_\mu\left[\left\{\frac{\partial^2}{\partial r_\lambda^2}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho_\lambda+\frac{a'}{a}\left\{\frac{\partial^2}{\partial r_\lambda\,\partial r'_\mu}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho'_\mu+\frac{1}{a}\left\{\frac{\partial^2}{\partial r_\lambda\,\partial\theta_\lambda}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta'_\mu)\right],\\
\ddot\eta_\lambda+2\omega\dot\rho_\lambda &= \frac{1}{a}\sum_\mu{}'\left[\left\{\frac{\partial^2}{\partial r_\lambda\,\partial\theta_\lambda}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0\rho_\lambda+\left\{\frac{\partial^2}{\partial\theta_\lambda\,\partial r_\mu}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0\rho_\mu+\frac{1}{a}\left\{\frac{\partial^2}{\partial\theta_\lambda^2}\left(\frac{m}{\Delta_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta_\mu)\right]\\
&\quad+\frac{1}{a}\sum_\mu\left[\left\{\frac{\partial^2}{\partial\theta_\lambda\,\partial r_\lambda}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho_\lambda+\frac{a}{a'}\left\{\frac{\partial^2}{\partial\theta_\lambda\,\partial r'_\mu}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho'_\mu+\frac{1}{a}\left\{\frac{\partial^2}{\partial\theta_\lambda^2}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta'_\mu)\right],\\
\ddot\rho'_\lambda-2\omega'\dot\eta'_\lambda-3\omega'^2\rho'_\lambda &= \sum_\mu{}'\left[\left\{\frac{\partial^2}{\partial r'^2_\lambda}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0\rho'_\lambda+\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial r'_\mu}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0\rho'_\mu+\frac{1}{a'}\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial\theta'_\lambda}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0(\eta'_\lambda-\eta'_\mu)\right]\\
&\quad+\sum_\mu\left[\frac{a}{a'}\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial r_\mu}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho_\mu+\left\{\frac{\partial^2}{\partial r'^2_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho'_\lambda+\frac{1}{a'}\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial\theta'_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0(\eta'_\lambda-\eta_\mu)\right],\\
\ddot\eta'_\lambda+2\omega'\dot\rho'_\lambda &= \frac{1}{a'}\sum_\mu{}'\left[\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial\theta'_\lambda}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0\rho'_\lambda+\left\{\frac{\partial^2}{\partial r'_\mu\,\partial\theta'_\lambda}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0\rho'_\mu+\frac{1}{a'}\left\{\frac{\partial^2}{\partial^2\theta'^2_\lambda}\left(\frac{m}{\Delta'_{\lambda\mu}}\right)\right\}_0(\eta'_\lambda-\eta'_\mu)\right]\\
&\quad+\frac{1}{a'}\sum_\mu\left[\frac{a}{a'}\left\{\frac{\partial^2}{\partial\theta'_\lambda\,\partial r_\mu}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho_\mu+\left\{\frac{\partial^2}{\partial\theta'_\lambda\,\partial r'_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho'_\lambda+\frac{1}{a'}\left\{\frac{\partial^2}{\partial\theta'^2_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0(\eta'_\lambda-\eta_\mu)\right].
\end{aligned}$$

In these equations, it will be remembered that λ and μ take all integral values from 1 to p, and the summation sign $\sum_\mu{}'$ omits the value $\mu=\lambda$.

Apply now the regular Poincaré transformation

$$k_s=\frac{1}{p}\sum_{\lambda=1}^{p}\rho_\lambda\,e^{-2\pi is\lambda/p},\qquad l_s=\frac{1}{p}\sum_{\lambda=1}^{p}\eta_\lambda\,e^{-2\pi is\lambda/p},\tag{9}$$

with the conjugate forms

$$\rho_\lambda=\sum_{s=1}^{p}k_s\,e^{2\pi is\lambda/p},\qquad \eta_\lambda=\sum_{s=1}^{p}l_s\,e^{2\pi is\lambda/p}\qquad (s=1,2,3,\ldots,p).\tag{10}$$

Similar forms are used for ρ', η'.

On multiplying each of the equations (8) by $(1/p)\,e^{-2\pi is\lambda/p}$ and summing from $\lambda=1$ to p, the left-hand members become

$$\begin{aligned}
&\ddot k_s-2\omega\dot l_s-3\omega^2k_s,\\
&\ddot l_s+2\omega\dot k_s,\\
&\ddot k'_s-2\omega'\dot l'_s-3\omega'^2k'_s,\\
&\ddot l'_s+2\omega'\dot k'_s.
\end{aligned}$$

The first three terms of the right-hand members are those that appear in the theory of a single ring. The results of the summation can be quoted from Pendse (1935). We retain only the significant terms. In the order of the equations the terms are

$$\frac{m}{a^3}\sum_{n=1}^{p-1}\left[-\frac{\sin^2(\pi ns/p)\,k_s}{4\sin^3(n\pi/p)}+\frac{\sin(2\pi ns/p)\cos(n\pi/p)\,il_s}{8\sin^2(n\pi/p)}\right],$$

$$\frac{m}{a^3}\sum_{n=1}^{p-1}\left[-\frac{\sin(2\pi ns/p)\cos(n\pi/p)\,ik_s}{8\sin^2(n\pi/p)}+\frac{\sin^2(ns\pi/p)\,l_s}{2\sin^3(n\pi/p)}\right],$$

$$\frac{m}{a'^3}\sum_{n=1}^{p-1}\left[-\frac{\sin^2(\pi ns/p)\,k'_s}{4\sin^3(n\pi/p)}+\frac{\sin(2\pi ns/p)\cos(n\pi/p)\,il'_s}{8\sin^2(n\pi/p)}\right],$$

$$\frac{m}{a'^3}\sum_{n=1}^{p-1}\left[-\frac{\sin(2\pi ns/p)\cos(n\pi/p)\,ik'_s}{8\sin^2(n\pi/p)}+\frac{\sin^2(ns\pi/p)\,l'_s}{2\sin^3(n\pi/p)}\right].$$

Putting

$$P_s=\sum_{n=1}^{p-1}\frac{\sin^2(\pi ns/p)}{4\sin^3(n\pi/p)},$$

$$Q_s=\sum_{n=1}^{p-1}\frac{\sin(2\pi ns/p)\cos(n\pi/p)}{8\sin^2(n\pi/p)},$$

the above terms become

$$\frac{m}{a^3}(-P_sk_s+iQ_sl_s),$$

$$\frac{m}{a^3}(-iQ_sk_s+2P_sl_s),$$

$$\frac{m}{a'^3}(-P_sk'_s+iQ_sl'_s),$$

$$\frac{m}{a'^3}(-iQ_sk'_s+2P_sl'_s).$$

The reduction of the remaining terms of equations (8) is more complicated. One example will be worked out and the results of the others quoted. Since

$$D_{\lambda\mu}^{-1}=\{r_\lambda^2+r_\mu'^2-2r_\lambda r'_\mu\cos(\theta_\lambda-\theta'_\mu)\}^{-\frac{1}{2}},$$

therefore

$$\begin{aligned}(D_{\lambda\mu}^{-1})_0&=\{a^2+a'^2-2aa'\cos\phi_{\lambda\mu}\}^{-\frac{1}{2}}\\&=(a')^{-1}\{1+\alpha^2-2\alpha\cos\phi_{\lambda\mu}\}^{-\frac{1}{2}}\\&=(a')^{-1}\left\{\tfrac{1}{2}b_0+\sum_{n-1}^{\infty}b_n\cos n\phi_{\lambda\mu}\right\}\quad(\alpha=a/a').\end{aligned}$$

Consider, for example, the fifth term of the right-hand member of the first equation (8):

$$\begin{aligned}\sum_\mu\frac{a'}{a}\left\{\frac{\partial^2}{\partial r_\lambda\partial r'_\mu}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho'_\mu&=\sum_\mu\frac{a'}{a}\left\{\frac{\partial^2}{\partial a\,\partial a'}\left(\frac{m}{D_{\lambda\mu}}\right)_0\right\}\rho'_\mu\\&=-\frac{m}{aa'^2}\sum_\mu\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)(\tfrac{1}{2}b_0+\sum b_n\cos n\phi_{\lambda\mu})\,\rho'_\mu.\end{aligned}$$

On substituting for ρ'_μ from (10) the expression becomes

$$-\frac{m}{aa'^2}\sum_\mu\sum_r\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)(\tfrac{1}{2}b_0+\sum b_n\cos n\phi_{\lambda\mu})\,k'_r\,e^{2\pi ir\mu/p}.$$

Now
$$\sum_{\mu}\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)b_0\,e^{2\pi ir\mu/p}=0;$$
and
$$\sum_n\sum_\mu\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)b_n\cos n\phi_{\lambda\mu}\,e^{2\pi ir\mu/p}$$
$$=\tfrac{1}{2}\sum_n\sum_\mu\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)b_n[\exp i\{n(\omega-\omega')\,t+2\pi n(\lambda-\mu)/p-n\epsilon+2\pi r\mu/p\}$$
$$+\exp i\{-n(\omega-\omega')\,t-2\pi n(\lambda-\mu)/p+n\epsilon+2\pi r\mu/p\}].$$

If we take it that terms in b of higher order than b_p are negligible, then the μ-summation of the above gives zero except when $n=r$, or $n+r=p$. In these cases the summation gives
$$-\frac{pm}{2aa'^2}\sum_r k'_r\left(2\frac{d}{d\alpha}+\alpha\frac{d^2}{d\alpha^2}\right)[b_r\exp i\{r(\omega-\omega')\,t+2\pi r\lambda/p-r\epsilon\}$$
$$+b_{p-r}\exp i\{-(p-r)\,(\omega-\omega')\,t+2\pi r\lambda/p+(p-r)\,\epsilon\}].$$

This has now to be multiplied by $(1/p)\;e^{-2\pi is\lambda/p}$, as before indicated, and the sum with regard to λ taken. The result is then
$$-\frac{m}{2aa'^2}\sum_\lambda\sum_r k'_r\left(2\frac{d}{d\alpha}+\frac{d^2}{d\alpha^2}\right)[b_r\exp i\{r(\omega-\omega')\,t+2\pi(r-s)\,\lambda/p-r\epsilon\}$$
$$+b_{p-r}\exp i\{-(p-r)\,(\omega-\omega')\,t+2\pi(r-s)\,\lambda/p+(p-r)\,\epsilon\}].$$

The λ-summation vanishes except when $r=s$, and the value is then
$$-\frac{pm}{2aa'^2}k'_s\left(2\frac{d}{d\alpha}+\frac{d^2}{d\alpha^2}\right)[b_s\exp i\{s(\omega-\omega')\,t-s\epsilon\}+b_{p-s}\exp i\{-(p-s)\,(\omega-\omega')\,t+(p-s)\,\epsilon\}].$$

The results for the remaining terms of the equations (8) are then as follows:
$$\sum_\mu\left\{\frac{\partial^2}{\partial r_\lambda^2}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho_\lambda=\frac{mp}{2a'^3}\frac{d^2b_0}{d\alpha^2}k_s,$$
$$\sum_\mu\frac{a'}{a}\left\{\frac{\partial^2}{\partial r_\lambda\,\partial r'_\mu}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho'_\mu=-\frac{mpk'_s}{2\alpha^2a'^3}\frac{d}{d\alpha}\alpha^2\frac{d}{d\alpha}[b_s\exp is\{(\omega-\omega')\,t-\epsilon\}$$
$$+b_{p-s}\exp i(p-s)\,\{-(\omega-\omega')\,t+\epsilon\}],$$
$$\sum_\mu\frac{1}{a}\left\{\frac{\partial^2}{\partial r_\lambda\,\partial\theta_\lambda}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta'_\mu)=-\frac{mpil'_s}{2\alpha a'^3}\frac{d}{d\alpha}[sb_s\exp is\{(\omega-\omega')\,t-\epsilon\}$$
$$-(p-s)\,b_{p-s}\exp i(p-s)\,\{-(\omega-\omega')t+\epsilon\}],$$
$$\sum_\mu\frac{1}{a}\left\{\frac{\partial^2}{\partial\theta_\lambda\,\partial r_\lambda}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho_\lambda=0,$$
$$\sum_\mu\frac{a'}{a^2}\left\{\frac{\partial^2}{\partial\theta_\lambda\,\partial r'_\mu}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0\rho'_\mu=-\frac{mpik'_s}{2\alpha^2a'^3}\frac{d}{d\alpha}[\alpha sb_s\exp is\{(\omega-\omega')\,t-\epsilon\}$$
$$-\alpha(p-s)\,b_{p-s}\exp i(p-s)\,\{-(\omega-\omega')\,t+\epsilon\}],$$
$$\sum_\mu\frac{1}{a^2}\left\{\frac{\partial^2}{\partial\theta_\lambda^2}\left(\frac{m}{D_{\lambda\mu}}\right)\right\}_0(\eta_\lambda-\eta'_\mu)=\frac{mpl'_s}{2a'^3\alpha^2}[s^2b_s\exp is\{(\omega-\omega')\,t-\epsilon\}$$
$$+(p-s)^2\,b_{p-s}\exp i(p-s)\,\{-(\omega-\omega')\,t+\epsilon\}],$$
$$\sum_\mu\frac{a}{a'}\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial r_\mu}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho_\mu=-\frac{mpk_s}{2a'^3}\frac{d}{d\alpha}\alpha^2\frac{d}{d\alpha}[b_s\exp is\{(\omega'-\omega)\,t+\epsilon\}$$
$$+b_{p-s}\exp i(p-s)\,\{-(\omega'-\omega)\,t-\epsilon\}],$$
$$\sum_\mu\left\{\frac{\partial^2}{\partial r'^2_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho'_\lambda=\frac{mpk'_s}{a'^3}\left(b_0+2\alpha\frac{db_0}{d\alpha}+\tfrac{1}{2}\alpha^2\frac{d^2b_0}{d\alpha^2}\right),$$

$$\frac{1}{a'}\sum_{\mu}\left\{\frac{\partial^2}{\partial r'_\lambda\,\partial\theta'_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0(\eta'_\lambda-\eta_\lambda)=\frac{mpil_s}{2a'^3}\frac{d}{d\alpha}[\alpha s b_s\exp is\{(\omega'-\omega)\,t+\epsilon\}$$
$$-\alpha(p-s)\,b_{p-s}\exp i(p-s)\,\{-(\omega'-\omega)\,t-\epsilon\}],$$

$$\sum_{\mu}\frac{a}{a'^2}\left\{\frac{\partial^2}{\partial\theta'_\lambda\,\partial r_\mu}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho_\mu=\frac{mpik_s}{2a'^3}\alpha\frac{d}{d\alpha}[sb_s\exp is\{(\omega'-\omega)\,t+\epsilon\}$$
$$-(p-s)\,b_{p-s}\exp i(p-s)\,\{-(\omega'-\omega)\,t-\epsilon\}],$$

$$\sum_{\mu}\frac{1}{a'}\left\{\frac{\partial^2}{\partial\theta'_\lambda\,\partial r'_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0\rho'_\lambda=0,$$

$$\sum_{\mu}\frac{1}{a'^2}\left\{\frac{\partial^2}{\partial\theta'^2_\lambda}\left(\frac{m}{D_{\mu\lambda}}\right)\right\}_0(\eta'_\lambda-\eta_\mu)=\frac{mpl_s}{2a'^3}[s^2b_s\exp is\{(\omega'-\omega)\,t+\epsilon\}$$
$$+(p-s)^2\,b_{p-s}\exp i(p-s)\,\{-(\omega'-\omega)\,t-\epsilon\}].$$

For short write

$$\tfrac{1}{2}pb_s=B_s,\qquad \tfrac{1}{2}p\frac{d}{d\alpha}(\alpha b_s)=C_s,$$

$$\tfrac{1}{2}p\frac{db_s}{d\alpha}=B'_s,\qquad \tfrac{1}{2}p\frac{d}{d\alpha}\left(\alpha^2\frac{db_s}{d\alpha}\right)=C'_s,$$

$$\tfrac{1}{2}p\frac{d^2b_s}{d\alpha^2}=B''_s,\quad p\left(b_0+2\alpha\frac{db_0}{d\alpha}+\tfrac{1}{2}\alpha^2\frac{d^2b_0}{d\alpha^2}\right)=D_0.$$

Since b_s and its derivatives are all positive, each of the quantities B, C and D is also positive.

If
$$\epsilon=m/M,\quad m/a'^3=\epsilon M/a'^3=\epsilon\omega'^2,$$
and
$$m/a'^3=\epsilon M\alpha^3/a^3=\epsilon\alpha^3\omega^2.$$

Let
$$\Omega=\frac{\omega}{\omega-\omega'},$$
$$\Omega'=\frac{\omega'}{\omega-\omega'},$$
$$(\omega-\omega')\,t-\epsilon=\tau,$$
and let the dots now indicate differentiation with regard to τ.

The equations (8) then become

$$\left.\begin{aligned}
\ddot{k}_s-2\Omega\dot{l}_s-3\Omega^2k_s&=\epsilon\Omega^2[k_s(\alpha^3B''_0-P_s)+il_sQ_s-k'_s\alpha\{C'_s\,e^{is\tau}+C'_{p-s}\,e^{-i(p-s)\tau}\}\\
&\qquad-il'_s\alpha^2\{sB'_s\,e^{is\tau}-(p-s)\,B'_{p-s}\,e^{-i(p-s)\tau}\}],\\
\ddot{l}_s+2\Omega\dot{k}_s&=\epsilon\Omega^2[-ik_sQ_s+2l_sP_s-ik'_s\alpha\{sC_s\,e^{is\tau}-(p-s)\,C_{p-s}\,e^{-i(p-s)\tau}\}\\
&\qquad+l'_s\alpha\{s^2B_s\,e^{is\tau}+(p-s)^2\,B_{p-s}\,e^{-i(p-s)\tau}\}],\\
\ddot{k}'_s-2\Omega'\dot{l}'_s-3\Omega'^2k'_s&=\epsilon\Omega'^2[k'_s(D_0-P_s)+il'_sQ_s-k_s\{C'_s\,e^{-is\tau}+C'_{p-s}\,e^{i(p-s)\tau}\}\\
&\qquad+il_s\{sC_s\,e^{-is\tau}-(p-s)\,C_{p-s}\,e^{i(p-s)\tau}\},\\
\ddot{l}'_s+2\Omega'\dot{k}'_s&=\epsilon\Omega'^2[-ik'_sQ_s+2l'_sP+ik_s\alpha\{sB'_s\,e^{-is\tau}-(p-s)\,B'_{p-s}\,e^{i(p-s)\tau}\}\\
&\qquad+l_s\{s^2B_s\,e^{-is\tau}+(p-s)^2\,B_{p-s}\,e^{i(p-s)\tau}\}].
\end{aligned}\right\}\quad(11)$$

These equations are linear, with periodic coefficients. The solutions are obtainable by the special methods appropriate to such equations.

Solution of the equations of stability

The equations (11) are linear with coefficients of period 2π in the independent variable τ. For convenience in writing we shall drop the suffix s, except where necessary, and restore it again later. By Flocquet's theorem, the solutions of this type of equation have the form

$$\left.\begin{aligned} k &= e^{\kappa\tau}f(\tau), \quad k' = e^{\kappa\tau}f'(\tau), \\ l &= e^{\kappa\tau}g(\tau), \quad l' = e^{\kappa\tau}g'(\tau), \end{aligned}\right\} \tag{12}$$

where f, g, f', g' are functions of τ of period 2π, and κ is a constant determined by the fulfilling of this condition. If κ is real and positive, it is evident that the solutions of the equations (11) will increase without limit as τ increases; and if κ is a pure imaginary, the solutions may be regarded as bounded and the system corresponding to them stable.

Since the equations involve a small parameter ϵ, we may apply Poincaré's theorem (1892), which states that the indices κ, and the functions f, f', g, g' may be expressed in power series of $\epsilon^{\frac{1}{2}}$, or ϵ, according to the nature of the roots of the characteristic equation. In the expansions of f, f', g, g' the arguments must be periodic functions of τ of period 2π.

It is shown by Poincaré that the series formed in this way are convergent for sufficiently small values of ϵ. We shall assume that this condition is satisfied.

The characteristic equation is obtained by putting $\epsilon = 0$ in (11) and solving. If a solution of the reduced equations is $e^{\kappa\tau}$, the equation to determine κ (the characteristic equation) is

$$\begin{vmatrix} \kappa^2 - 3\Omega^2, & 2\Omega\kappa, & \cdot & \cdot \\ 2\Omega\kappa, & \kappa^2, & \cdot & \cdot \\ \cdot & \cdot & \kappa^2 - 3\Omega'^2, & 2\Omega'\kappa \\ \cdot & \cdot & 2\Omega'\kappa, & \kappa^2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \kappa^2 - 3\Omega^2, & 2\Omega\kappa \\ 2\Omega\kappa, & \kappa^2 \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} \kappa^2 - 3\Omega'^2, & 2\Omega'\kappa \\ 2\Omega'\kappa, & \kappa^2 \end{vmatrix} = 0.$$

Hence the roots of the characteristic equation are $(\pm i\Omega, 0, 0)$ and $(\pm i\Omega', 0, 0)$.

The solutions for the zero roots are obtained in series of integral powers of $\epsilon^{\frac{1}{2}}$; and since $\Omega - \Omega' = 1$, the remaining roots occur in pairs which differ by an imaginary integer, from which it follows that the series involved are in integral powers of ϵ.

A special case would arise if Ω should be an integer n. The equation to determine n in terms of the data of the problem is

$$\left(\frac{1 - \nu\pi/p}{1 + \nu\pi/p}\right)^3 = \left(1 - \frac{1}{n}\right)^2.$$

An integral solution for n would require a special adjustment of ν and p. We shall therefore assume Ω to be non-integral.

It should be noted that the complete solution of equations for given s must exhibit eight linearly independent integrals.

Substitute now the form (12) in the equations (11). Then we have

$$\left.\begin{aligned}
&\kappa^2 f + 2\kappa\dot{f} + \ddot{f} - 2\Omega(\kappa g + \dot{g}) - 3\Omega^2 f \\
&\qquad = \epsilon\Omega^2[f(\alpha^3 B_0'' - P) + igQ - f'\alpha\{C_s' \,\mathrm{e}^{is\tau} + C_{p-s}' \,\mathrm{e}^{-i(p-s)\tau}\} \\
&\qquad\qquad - ig'\alpha^2\{sB_s' \,\mathrm{e}^{is\tau} - (p-s)\, B_{p-s}' \,\mathrm{e}^{-i(p-s)\tau}\}], \\
&\kappa^2 g + 2\kappa\dot{g} + \ddot{g} + 2\Omega(\kappa f + \dot{f}) = \epsilon\Omega^2[-ifQ + 2gP - if'\alpha\{sC_s \,\mathrm{e}^{is\tau} - (p-s)\, C_{p-s} \,\mathrm{e}^{-i(p-s)\tau}\} \\
&\qquad\qquad + g'\alpha\{s^2 B_s \,\mathrm{e}^{is\tau} + (p-s)^2\, B_{p-s} \,\mathrm{e}^{-i(p-s)\tau}\}], \\
&\kappa^2 f' + 2\kappa\dot{f}' + \ddot{f}' - 2\Omega'(\kappa g' + \dot{g}') - 3\Omega'^2 f' \\
&\qquad = \epsilon\Omega'^2[f'(D_0 - P) + ig'Q - f\{C_s' \,\mathrm{e}^{-is\tau} + C_{p-s}' \,\mathrm{e}^{i(p-s)\tau}\} \\
&\qquad\qquad + ig\{sC_s \,\mathrm{e}^{-is\tau} - (p-s)\, C_{p-s} \,\mathrm{e}^{i(p-s)\tau}\}], \\
&\kappa^2 g' + 2\kappa\dot{g}' + \ddot{g}' + 2\Omega'(\kappa f' + \dot{f}') \\
&\qquad = \epsilon\Omega'^2[-if'Q + 2g'P + if\alpha\{sB_s' \,\mathrm{e}^{-is\tau} - (p-s)\, B_{p-s}' \,\mathrm{e}^{i(p-s)\tau}\} \\
&\qquad\qquad + g\{s^2 B_s \,\mathrm{e}^{-is\tau} + (p-s)^2\, B_{p-s} \,\mathrm{e}^{i(p-s)\tau}\}].
\end{aligned}\right\} \quad (13)$$

Solutions for the cases where the characteristics differ by an imaginary integer

Let

$$f = f_0 + \epsilon f_1 + \ldots, \quad f' = f_0' + \epsilon f_1' + \ldots,$$
$$g = g_0 + \epsilon g_1 + \ldots, \quad g' = g_0' + \epsilon g_1' + \ldots,$$
$$\kappa = \kappa_0 + \epsilon\kappa_1 + \ldots,$$

Substitute in equations (13) and equate to zero the coefficients of corresponding powers of ϵ. We consider the two pairs of characteristics separately.

$$\text{(i)} \quad \kappa_0 = i\Omega, \quad \Omega - \Omega' = 1$$

Consider the terms independent of ϵ.

The equations resulting are

$$\left.\begin{aligned}
&-4\Omega^2 f_0 + 2i\Omega\dot{f}_0 + \ddot{f}_0 - 2\Omega^2 i g_0 - 2\Omega\dot{g}_0 &&= 0, \\
&-\Omega^2 g_0 + 2i\Omega\dot{g}_0 + \ddot{g}_0 + 2\Omega^2 i f_0 + 2\Omega\dot{f}_0 &&= 0, \\
&-\Omega^2 f_0' + 2i\Omega\dot{f}_0' + \ddot{f}_0' - 2\Omega\Omega' i g_0' - 2\Omega'\dot{g}_0' - 3\Omega'^2 f_0' &&= 0, \\
&-\Omega^2 g_0' + 2i\Omega\dot{g}_0' + \ddot{g}_0' + 2\Omega\Omega' i f_0' + 2\Omega'\dot{f}_0' &&= 0.
\end{aligned}\right\} \quad (14)$$

These equations form two independent pairs. The solutions are: for the first pair, 1, $\mathrm{e}^{-i\Omega\tau}$, $\mathrm{e}^{-i\Omega\tau}$, $\mathrm{e}^{-2i\Omega\tau}$; and for the second pair, $\mathrm{e}^{-i\Omega\tau}$, $\mathrm{e}^{-i\Omega\tau}$, $\mathrm{e}^{-i\tau}$, $\mathrm{e}^{-i(2\Omega-1)\tau}$. As already stated, the functions f, f', g, g' are to be determined as having the period 2π in τ (or to be constants). Hence the solutions fulfilling this condition are

$$\left.\begin{aligned}
f_0 &= \bar{f}_0, & f_0' &= \bar{f}_0' \,\mathrm{e}^{-i\tau}, \\
\dot{g}_0 &= 2i\bar{f}_0, & g_0' &= 2i\bar{f}_0' \,\mathrm{e}^{-i\tau},
\end{aligned}\right\} \quad (15)$$

where $\bar{f}_0, \bar{f}_0'$ are arbitrary constants.

Consider next the terms associated with the first power of ϵ. The resulting equations are, on inserting the values for f_0, g_0, f_0', g_0' as given in (15),

$$\left.\begin{aligned}
&-4\Omega^2 f_1 + 2i\Omega\dot{f}_1 + \ddot{f}_1 - 2i\Omega^2 g_1 - 2\Omega\dot{g}_1\\
&\qquad = \{2i\Omega\kappa_1 + \Omega^2(\alpha^3 B_0'' - P - 2Q)\}\bar{f}_0\\
&\qquad\quad - \Omega^2[(\alpha C_s' - 2\alpha^2 s B_s')\,\mathrm{e}^{is\tau} + \{\alpha C_{p-s}' + 2\alpha^2(p-s)\,B_{p-s}'\}\,\mathrm{e}^{-i(p-s)\tau}]\bar{f}_0'\,\mathrm{e}^{-i\tau},\\
&-\Omega^2 g_1 + 2i\Omega\dot{g}_1 + \ddot{g}_1 + 2\Omega^2 i f_1 + 2\Omega\dot{f}_1\\
&\qquad = \{2\Omega\kappa_1 + \Omega^2 i(4P - Q)\}\bar{f}_0\\
&\qquad\quad - \Omega^2 i[(\alpha s C_s - 2\alpha s^2 B_s)\,\mathrm{e}^{is\tau} - \{\alpha(p-s)\,C_{p-s} - 2\alpha(p-s)^2\,B_{p-s}\}\,\mathrm{e}^{-i(p-s)\tau}]\bar{f}_0'\,\mathrm{e}^{-i\tau},\\
&-(\Omega^2 + 3\Omega'^2)f_1' + 2i\Omega\dot{f}_1' + \ddot{f}_1' - 2i\Omega\Omega' g_1' - 2\Omega'\dot{g}_1'\\
&\qquad = \{2i\Omega'\kappa_1 + \Omega'^2(D_0 - P - 2Q)\}\bar{f}_0'\,\mathrm{e}^{-i\tau}\\
&\qquad\quad + \Omega'^2[-(C_s' + 2sC_s)\,\mathrm{e}^{is\tau} - \{C_{p-s}' - 2(p-s)\,C_{p-s}\}\,\mathrm{e}^{i(p-s)\tau}]\bar{f}_0,\\
&-\Omega^2 g_1' + 2i\Omega\dot{g}_1' + \ddot{g}_1' + 2i\Omega\Omega' f_1' + 2\Omega'\dot{f}_1'\\
&\qquad = \{2\Omega'\kappa_1 + \Omega'^2 i(4P - Q)\}\bar{f}_0'\,\mathrm{e}^{-i\tau}\\
&\qquad\quad + \Omega'^2 i[(\alpha s B_s' + 2\alpha s^2 B_s)\,\mathrm{e}^{-is\tau} + \{-(p-s)\,\alpha B_{p-s}' + 2(p-s)^2\,\alpha B_{p-s}\}\,\mathrm{e}^{i(p-s)\tau}]\bar{f}_0.
\end{aligned}\right\} \quad (16)$$

The left-hand members of these equations have the same form as those of (14), and the complementary functions will be the same in form as (15) with fresh arbitrary constants.

The right-hand members of these equations consist of constant terms and terms of period 2π in τ. The occurrence of constant terms in the right-hand members of the first pair will give rise on integration to terms of the form $K\tau$, K being a constant. And the occurrence of terms in $\mathrm{e}^{-i\tau}$ in the right-hand members of the second pair will give rise to terms of the form $\tau\,\mathrm{e}^{-i\tau}$. The undetermined constant κ_1 and the constants of integration must be chosen so that such terms disappear from the integrals.

Two cases arise which require separate consideration: $s = 1$, $s \neq 1$. The case $s = p+1$ cannot arise.

(*a*) $s \neq 1$

If we deal with the critical terms only, the equations (16) again break up into two independent pairs. The first pair gives

$$\left.\begin{aligned}
-4\Omega^2 f_1 + 2i\Omega\dot{f}_1 + \ddot{f}_1 - 2\Omega^2 i g_1 - 2\Omega\dot{g}_1 &= \{2i\Omega\kappa_1 + \Omega^2(\alpha^3 B_0'' - P - 2Q)\}\bar{f}_0,\\
-\Omega^2 g_1 + 2i\Omega\dot{g}_1 + \ddot{g}_1 + 2\Omega^2 i f_1 + 2\Omega\dot{f}_1 &= \{2\Omega\kappa_1 + \Omega^2 i(4P - Q)\}\bar{f}_0.
\end{aligned}\right\} \quad (17)$$

In order to avoid non-periodic terms arising in the particular integral we must have

$$\bar{f}_0[-2i\Omega\kappa_1 + \Omega^2(\alpha^3 B_0'' + 7P - 4Q)] = 0.$$

Hence, either $$\bar{f}_0 = 0$$

or $$2i\kappa_1 = \Omega(\alpha^3 B_0'' + 7P - 4Q) \quad (18)$$

and $\bar{f}_0$ is arbitrary.

Since the right-hand member of (18) is real, the value of κ_1 is a pure imaginary.

Next, take the critical terms of the right-hand members of the second pair of equations (16). We then have

$$\left.\begin{aligned}
-(\Omega^2 + 3\Omega'^2)f_1' + 2i\Omega\dot{f}_1' + \ddot{f}_1' - 2i\Omega\Omega' g_1' - 2\Omega'\dot{g}_1' &= \{2i\Omega'\kappa_1 + \Omega'^2(D_0 - P - 2Q)\}\bar{f}_0'\,\mathrm{e}^{-i\tau},\\
-\Omega^2 g_1' + 2i\Omega\dot{g}_1' + \ddot{g}_1' + 2i\Omega\Omega' f_1' + 2\Omega'\dot{f}_1' &= \{2\Omega'\kappa + \Omega'^2 i(4P - Q)\}\bar{f}_0'\,\mathrm{e}^{-i\tau}.
\end{aligned}\right\} \quad (19)$$

In order to avoid the appearance of terms of the form $\tau\,\mathrm{e}^{-i\tau}$ in the particular integral we must have

$$\bar{f}_0'\{-2i\Omega'\kappa_1+\Omega'^2(D_0+7P-4Q)\}=0.$$

Hence either

$$\bar{f}_0'=0,$$

or

$$2i\kappa_1=\Omega'(D_0+7P-4Q), \tag{20}$$

and $\bar{f}_0$ is arbitrary.

Since the right-hand member of (20) is real, κ_1 is again a pure imaginary.

When the remaining terms of the right-hand members of equations (16) are included, the corresponding particular integrals will be periodic. To verify this consider a term $\mathrm{e}^{i\sigma\tau}$ placed in the right-hand side of equations (17), σ being an integer including zero. The corresponding particular integral will have the denominator

$$\begin{vmatrix} (\sigma+\Omega)^2+3\Omega^2, & 2i\Omega(\sigma+\Omega) \\ 2i\Omega(\sigma+\Omega), & -(\sigma+\Omega)^2 \end{vmatrix}.$$

This determinant vanishes only for $\sigma=0,\ -\Omega,\ -2\Omega$. As Ω is not an integer, and the case $\sigma=0$ has already been considered, all the other particular integrals are finite, determinate and periodic.

Similarly, for the equations (17), the determinant is

$$\begin{vmatrix} (\sigma+\Omega)^2+3\Omega'^2, & 2i\Omega'(\sigma+\Omega) \\ 2i\Omega'(\sigma+\Omega), & -(\sigma+\Omega)^2 \end{vmatrix}.$$

This determinant vanishes for $\sigma=-\Omega,\ -2\Omega+1,\ -1$.

The case $\sigma=-1$ is the case just considered and the other roots are non-integral.

(*b*) $s=1$

Take now equations (16) with $s=1$. As we are chiefly interested in the nature of the constant κ_1, we shall write down only the critical terms of the right-hand member. The equations then become

$$\left.\begin{aligned}
&-4\Omega^2 f_1+2i\Omega\dot{f}_1+\ddot{f}_1-2\Omega^2 ig_1-2\Omega\dot{g}_1\\
&\qquad=\{2i\Omega\kappa_1+\Omega^2(\alpha^3B_0''-P_1-Q_1)\}\bar{f}_0-\Omega^2(\alpha C_1'-2\alpha^2B_1')\bar{f}_0',\\
&-\Omega^2 g_1+2i\Omega\dot{g}_1+\ddot{g}_1+2\Omega^2 if_1+2\Omega\dot{f}_1\\
&\qquad=\{2\Omega\kappa_1+\Omega^2 i(4P_1-Q_1)\}\bar{f}_0-\Omega^2 i(\alpha C_1-2\alpha B_1)\bar{f}_0',\\
&-(\Omega^2+3\Omega'^2)f_1'+2i\Omega\dot{f}_1'+\ddot{f}_1'-2i\Omega\Omega' g_1'-2\Omega'\dot{g}_1'\\
&\qquad=\{2i\Omega'\kappa_1+\Omega'^2(D_0-P_1-2Q_1)\}\bar{f}_0'\,\mathrm{e}^{-i\tau}-\Omega'^2(C_1'+2C_1)\bar{f}_0\,\mathrm{e}^{-i\tau},\\
&-\Omega^2 g_1'+2i\Omega\dot{g}_1'+\ddot{g}_1'+2i\Omega\Omega' f_1'+2\Omega'\dot{f}_1'\\
&\qquad=\{2\Omega'\kappa_1+\Omega'^2 i(4P_1-Q_1)\}\bar{f}_0'\,\mathrm{e}^{-i\tau}+\Omega'^2 i(\alpha B_1'+2\alpha B_1)\bar{f}_0\,\mathrm{e}^{-i\tau}.
\end{aligned}\right\} \tag{21}$$

To remove the non-periodic terms which arise in the integration of these equations, we must have

$$\left.\begin{aligned}
&\bar{f}_0\{-2i\Omega\kappa_1+\Omega^2(\alpha^3B_0''+7P_1-4Q_1)\}+\bar{f}_0'\,\Omega^2\alpha(-C_1'+2\alpha B_1'-2C_1+4B_1)=0,\\
&\bar{f}_0\{-C_1'+2\alpha B_1'-2C_1+4B_1\}\,\Omega'^2+\bar{f}_0'\{-2i\Omega'\kappa_1+\Omega'^2(D_0+7P_1-4Q_1)\}=0.
\end{aligned}\right\} \tag{22}$$

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By definition
$$7P_1-4Q_1=\frac{7}{4}\sum_{n=1}^{p-1}\operatorname{cosec}(n\pi/p)-\sum_{n=1}^{p-1}\cos^2(n\pi/p)\operatorname{cosec}(n\pi/p)$$
$$=\frac{3}{4}\sum_{n=1}^{p-1}\operatorname{cosec}(n\pi/p)+\sum_{n=1}^{p-1}\sin(n\pi/p),$$
which is positive.

Also B_0'' and D_0 are essentially positive.

Putting $2i\kappa_1=x$, the determinant of the equations (22) may be written
$$\Delta\equiv\begin{vmatrix}-\Omega x+\gamma & \Omega^2\delta\alpha\\ \Omega'^2\delta & -\Omega' x+\gamma'\end{vmatrix}=0,$$
where γ, γ' are real and positive, and δ is real.

Then we have

$x=$	$-\infty$	γ'/Ω'	γ/Ω	$+\infty$
$\Delta=$	$+$	$-$	$-$	$+$

Hence Δ vanishes for a real value of x between $-\infty$ and the smaller of γ'/Ω', γ/Ω; and again for a real value of between the greater of γ'/Ω', γ/Ω and $+\infty$.

Hence κ_1 has two purely imaginary and distinct values.

For these determined values of κ_1, $\bar{f}_0'$ is expressed in terms of $\bar{f}_0$ by (22), thus introducing only one arbitrary constant in each case.

(ii) $\kappa_0=-i\Omega$

The terms independent of ϵ are
$$\left.\begin{aligned}&-4\Omega^2f_0-2i\Omega\dot{f}_0+\ddot{f}_0+2i\Omega^2g_0-2\Omega\dot{g}_0&&=0,\\&-\Omega^2g_0-2i\Omega\dot{g}_0+\ddot{g}_0-2i\Omega^2f_0+2\Omega\dot{f}_0&&=0,\\&-(\Omega^2+3\Omega'^2)f_0'-2i\Omega\dot{f}_0'+\ddot{f}_0'+2i\Omega\Omega'g_0'-2\Omega'\dot{g}_0'&&=0,\\&-\Omega^2g_0'-2i\Omega\dot{g}_0'+\ddot{g}_0'-2i\Omega\Omega'f_0'+2\Omega'\dot{f}_0'&&=0.\end{aligned}\right\}\tag{23}$$

The periodic solutions are
$$f_0=\bar{f}_0,\qquad f_0'=\bar{f}_0'\,e^{i\tau},$$
$$g_0=-2i\bar{f}_0,\quad g_0'=-2i\bar{f}_0'\,e^{i\tau},$$
$\bar{f}_0$, $\bar{f}_0'$ being arbitrary constants.

Taking now the terms factored by ϵ, two cases arise as before, according as s equals unity or not.

(*a*) $s\neq1$

For the first pair, corresponding to equations (17), we have
$$\left.\begin{aligned}-4\Omega^2f_1-2i\Omega\dot{f}_1+\ddot{f}_1+2\Omega^2ig_1-2\Omega\dot{g}_1&=\{-2i\Omega\kappa_1+\Omega^2(\alpha^3B_0''-P+2Q)\}\bar{f}_0,\\-\Omega^2g_1-2i\Omega\dot{g}_1+\ddot{g}_1-2i\Omega^2f_1+2\Omega\dot{f}_1&=\{2\Omega\kappa_1-\Omega^2i(Q+4P)\}\bar{f}_0.\end{aligned}\right\}\tag{24}$$

From these we deduce that to avoid non-periodic terms arising in the solution, we must have
$$\bar{f}_0\{2i\Omega\kappa_1+\Omega^2(\alpha^3B_0''+7P+4Q)\}=0.$$

Hence either
$$\bar{f}_0=0;$$
or
$$2i\Omega\kappa+\Omega^2(\alpha^3B_0''+7P+4Q)=0\tag{25}$$
and $\bar{f}_0$ is arbitrary.

Again κ_1 is a pure imaginary.

For the second pair, corresponding to equations (19), we have

$$-(\Omega^2+3\Omega'^2)f_1'-2i\Omega\dot{f}_1'+\ddot{f}_1'+2i\Omega\Omega' g_1'-2\Omega'\dot{g}_1' = \{-2i\Omega'\kappa_1+\Omega'^2(D_0-P+2Q)\}\bar{f}_0'\,e^{i\tau},$$
$$-\Omega^2 g_1'-2i\Omega\dot{g}_1'+\ddot{g}_1'-2i\Omega\Omega' f_1'+2\Omega'\dot{f}_1' = \{2\Omega'\kappa_1-\Omega'^2 i(4P+Q)\}\bar{f}_0'\,e^{i\tau}.$$

To avoid the occurrence of non-periodic terms we must have

$$\bar{f}_1'\{2i\Omega'\kappa_1+\Omega'^2(D_0+7P+4Q)\} = 0.$$

Hence again, either $\bar{f}_1' = 0$;

or $$2i\Omega'\kappa_1+\Omega'^2(D_0+7P+4Q) = 0, \tag{26}$$

and $\bar{f}_0'$ is arbitrary.

Also κ_1 is again a pure imaginary.

(*b*) $s = 1$

Again take equations corresponding to (21) and, as before, include only the critical terms. The resulting equations are

$$\left.\begin{aligned}
&-4\Omega^2 f_1-2i\Omega\dot{f}_1+\ddot{f}_1+2\Omega^2 i g_1-2\Omega\dot{g}_1\\
&\qquad = \{-2i\Omega\kappa_1+\Omega^2(\alpha^3 B_0''-P_1+2Q_1)\}\bar{f}_0-\Omega^2\alpha(C_1'-2\alpha B_1')\bar{f}_0',\\
&-\Omega^2 g_1-2i\Omega\dot{g}_1+\ddot{g}_1-2i\Omega^2 f_1+2\Omega\dot{f}_1\\
&\qquad = \{2\Omega\kappa_1-\Omega^2 i(Q_1+4P_1)\}\bar{f}_0+\Omega^2 i\alpha(C_1-2B_1)\bar{f}_0',\\
&-(\Omega^2+3\Omega'^2)f_1'-2i\Omega\dot{f}_1'+\ddot{f}_1'+2i\Omega\Omega' g_1'-2\Omega'\dot{g}_1'\\
&\qquad = \{-2i\Omega'\kappa_1+\Omega'^2(D_0-P_1+2Q_1)\bar{f}_0'\,e^{i\tau}-\Omega'^2(C_1'+2C_1)\bar{f}_0 e^{i\tau},\\
&-\Omega^2 g_1'-2i\Omega\dot{g}_1'+\ddot{g}_1'-2i\Omega\Omega' f_1'+2\Omega'\dot{f}_1'\\
&\qquad = \{2\Omega'\kappa_1-\Omega'^2 i(4P+Q)\}\bar{f}_0'\,e^{i\tau}-\Omega'^2 i(\alpha B_1'+2B_1)\bar{f}_0\,e^{i\tau}.
\end{aligned}\right\} \tag{27}$$

The condition that no non-periodic terms may appear in the particular integrals of these equations is

$$\left.\begin{aligned}
\bar{f}_0\{2i\Omega\kappa_1+\Omega^2(\alpha^3 B_0''+7P_1+4Q_1)\}+\bar{f}_0'\,\Omega^2\alpha(-C_1'-2C_1+2\alpha B_1'+4B_1) = 0,\\
\bar{f}_0\{-C_1'-2C_1+2\alpha B_1'+4B_1\}\,\Omega'^2+\bar{f}_0'\{2i\Omega'\kappa_1+\Omega'^2(D_0+7P_1+4Q_1)\} = 0.
\end{aligned}\right\} \tag{28}$$

An examination of the determinant of (28) shows again that $i\kappa_1$ has two real values.

Collecting up the results (18), (20), (22), (25), (26), (28), we see that there are four independent solutions in each of the two cases, $s = 1$, $s \neq 1$, and in each case the exponent κ_1 is a pure imaginary.

Solutions when the characteristics are 0, 0

In this case theory indicates that the solutions are in the form of series of powers of $\epsilon^{\frac{1}{2}}$. Therefore we take

$$f = f_0+\epsilon^{\frac{1}{2}}f_1+\epsilon f_2+\ldots, \quad f' = f_0'+\epsilon^{\frac{1}{2}}f_1'+\epsilon f_2'+\ldots,$$
$$g = g_0+\epsilon^{\frac{1}{2}}g_1+\epsilon g_2+\ldots, \quad g' = g_0'+\epsilon^{\frac{1}{2}}g_1'+\epsilon g_2'+\ldots,$$
$$\kappa = \epsilon^{\frac{1}{2}}\kappa_1+\epsilon\kappa_2+\ldots.$$

Substitute these values in equations (13) and equate to zero the coefficients of each power of ϵ.

The terms independent of ϵ give

$$\begin{aligned} \ddot{f}_0 - 2\Omega\dot{g}_0 - 3\Omega^2 f_0 &= 0, \\ \ddot{g}_0 + 2\Omega\dot{f}_0 &= 0, \\ \ddot{f}'_0 - 2\Omega'\dot{g}'_0 - 3\Omega'^2 f'_0 &= 0, \\ \ddot{g}'_0 + 2\Omega'\dot{f}'_0 &= 0. \end{aligned}$$

The only solutions of these which are periodic of period 2π (or constants) are

$$f_0 = 0, \qquad f'_0 = 0,$$
$$g_0 = \text{const.} = \bar{g}_0, \quad g'_0 = \text{const.} = \bar{g}'_0.$$

Take next the terms in $\epsilon^{\frac{1}{2}}$.

$$\begin{aligned} \ddot{f}_1 - 2\Omega\dot{g}_1 - 3\Omega^2 f_1 &= 2\Omega\kappa_1\bar{g}_0, \\ \ddot{g}_1 + 2\Omega\dot{f}_1 &= 0, \\ \ddot{f}'_1 - 2\Omega'\dot{g}'_1 - 3\Omega'^2 f'_1 &= 2\Omega'\kappa_1\bar{g}'_0, \\ \ddot{g}'_1 + 2\Omega'\dot{f}'_1 &= 0. \end{aligned}$$

In the same way from these we have

$$f_1 = -\frac{2\kappa_1\bar{g}_0}{3\Omega}, \qquad f'_1 = -\frac{2\kappa_1\bar{g}_0}{3\Omega'},$$
$$g_1 = \text{const.} = \bar{g}_1, \quad g'_1 = \text{const.} = \bar{g}'_1.$$

Here κ_1 remains as yet undetermined.

The terms factoring ϵ give

$$\left.\begin{aligned} \ddot{f}_2 - 2\Omega\dot{g}_2 - 3\Omega^2 f_2 &= 2\Omega\kappa_1\bar{g}_1 + 2\Omega\kappa_2\bar{g}_0 + \Omega^2[i\bar{g}_0\,Q_s - i\bar{g}'_0\alpha^2\{sB'_s\,e^{is\tau} - (p-s)\,B'_{p-s}\,e^{-i(p-s)\tau}\}], \\ \ddot{g}_2 + 2\Omega\dot{f}_2 &= \tfrac{1}{3}\kappa_1^2\bar{g}_0 + \Omega^2[2\bar{g}_0 P_s + \bar{g}'_0\alpha\{s^2B_s\,e^{is\tau} + (p-s)^2\,B_{p-s}\,e^{-i(p-s)\tau}\}], \\ \ddot{f}'_2 - 2\Omega'\dot{g}'_2 - 3\Omega'^2 f'_2 &= 2\Omega'\kappa_1\bar{g}'_1 + 2\Omega'\kappa_2\bar{g}'_0 + \Omega'^2[i\bar{g}'_0\,Q_s + i\bar{g}_0\{sC_s\,e^{-is\tau} - (p-s)\,C_{p-s}\,e^{i(p-s)\tau}\}], \\ \ddot{g}'_2 + 2\Omega'\dot{f}'_2 &= \tfrac{1}{3}\kappa_1^2\bar{g}'_0 + \Omega'^2[2\bar{g}'_0 P_s + \bar{g}_0\{s^2B_s\,e^{-is\tau} + (p-s)^2\,B_{p-s}\,e^{i(p-s)\tau}\}]. \end{aligned}\right\} \tag{29}$$

The second of equations (29) gives

$$\dot{g}_2 + 2\Omega f_2 = \{\tfrac{1}{3}\kappa_1^2\bar{g}_0 + 2\Omega^2 P_s\bar{g}_0\}\tau + \bar{g}_2 - is\alpha\Omega^2\bar{g}'_0 B_s\,e^{is\tau} + i(p-s)\,\alpha\Omega^2\bar{g}'_0 B_{p-s}\,e^{-i(p-s)\tau}, \tag{30}$$

$\bar{g}_2$ being an arbitrary constant.

The first of equations (29) then gives

$$\begin{aligned} \ddot{f}_2 + \Omega^2 f_2 = {} & 2\Omega\kappa_1\bar{g}_1 + 2\Omega\kappa_2\bar{g}_0 + \Omega^2[i\bar{g}_0\,Q_s - i\bar{g}'_0\alpha^2\{sB'_s\,e^{is\tau} - (p-s)\,B'_{p-s}\,e^{-i(p-s)\tau}\}] \\ & + 2\Omega(\tfrac{1}{3}\kappa_1^2\bar{g}_0 + 2\Omega^2 P_s\bar{g}_0)\,\tau + 2\Omega\bar{g}_2 - 2i\Omega^3\bar{g}'_0\{sB_s\,e^{is\tau} - (p-s)\,B_{p-s}\,e^{-i(p-s)\tau}\}. \end{aligned} \tag{31}$$

To avoid non-periodic terms appearing in the integral of (21) we must have

$$\tfrac{1}{3}\kappa_1^2\bar{g}_0 + 2\Omega^2 P_s\bar{g}_0 = 0.$$

Hence either

$$\bar{g}_0 = 0$$

or

$$\kappa_1 = \pm i\Omega(6P_s)^{\frac{1}{2}},$$

and $\bar{g}_0$ is arbitrary.

Since Ω, except in very special circumstances, is not an integer, the terms in $e^{is\tau}$ and $e^{-i(p-s)\tau}$ in the right member of (31) produce corresponding terms in the integral with non-vanishing denominators. The constant terms appearing in the same member, however, need attention. Corresponding to them we have the particular integral

$$\Omega^2 f_2 = 2\Omega\kappa_1\bar{g}_1 + 2\Omega\kappa_2\bar{g}_0 + 2\Omega\bar{g}_2 + i\Omega^2\bar{g}_0\,Q_s.$$

These terms substituted in (30) give

$$g_2 = -(3\bar{g}_2 + 4\kappa_1 \bar{g}_1 + 4\kappa_2 \bar{g}_0 + 2i\Omega Q_s \bar{g}_0)\,\tau + \text{const.}$$

Hence again, to avoid non-periodic terms, we must have

$$3\bar{g}_2 + 4\kappa_1 \bar{g}_1 + 4\kappa_2 \bar{g}_0 + 2i\Omega Q_s \bar{g}_0 = 0. \tag{32}$$

The constant κ_2 is determined at the next stage. Then (32) gives a relation between the arbitrary constants.

Consider next the third and fourth equations of (29). The form of the equations being similar to the first two of (29), the results may be written down at once. We have either

$$\bar{g}_0' = 0,$$

or

$$\kappa_1 = \pm i\Omega'(6P_s)^{\frac{1}{2}}$$

and $\bar{g}_0'$ arbitrary.

Also,

$$3\bar{g}_2' + 4\kappa_1 \bar{g}_1' + 4\kappa_2 \bar{g}_0' + 2i\Omega' Q_s \bar{g}_0' = 0.$$

We have therefore four solutions:

(i) $\bar{g}_0$ arbitrary, $\bar{g}_0' = 0$, $\kappa_1 = \pm i\Omega(6P_s)^{\frac{1}{2}}$;

(ii) $\bar{g}_0'$ arbitrary, $\bar{g}_0 = 0$, $\kappa_1 = \pm i\Omega'(6P_s)^{\frac{1}{2}}$.

Since P_s is essentially real and positive for all valid values of s, each of the solutions represents stability.

Conclusion

The solutions of the stability equations just found depend upon the possibility of convergent series in terms of powers of the parameter ϵ, or m/M. Poincaré's theory shows that the convergence of these series is ensured if ϵ is sufficiently small, though no satisfactory upper limit of its value is available. In the Saturnian system the ratio m/M, that of the mass of a single particle to that of Saturn must be exceedingly small, though again no estimate can be given except that indicated by Maxwell's theory, which we have assumed for the purposes of certain comparisons. The conclusion can be expressed that a pair of rings of the kind described in this paper will form a stable system if the masses of the component particles are small enough compared with that of the primary; and it seems likely that this condition is fulfilled in the Saturnian system.

In working out the series only terms as far as the first power of ϵ have been dealt with, and this would seem sufficient for the purpose.

The indications of the analysis lead to a fair inference that a system of a larger finite number of similar rings would also be stable.

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